Characterizing brace-minimal rigidity of square-grid frameworks with holes

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Abstract: The *rigidity* of *square-grid frameworks* was first studied by Bolker and Crapo [1]. They gave a combinatorial characterization for the rigidity of frameworks with no holes. After two decades, Gáspár, Radics and Recski [2] extended the result to frameworks with holes and provided a method to determine the rigidity faster than computing the rank of its *rigidity matrix*. While the characterization by Bolker et al. immediately provides a necessary and sufficient condition for the *brace-minimal rigidity* of frameworks with no holes, one by Gáspár et al. does not provide such an explicit condition for the case with holes. In this paper, we give the first necessary and sufficient condition for the brace-minimal rigidity of square-grid frameworks with holes.

Keywords: Combinatorial rigidity; Bar-joint framework; Square-grid framework; Bracing

1 Introduction

A *d-dimensional bar-joint framework* is a collection of one-dimensional rigid *bars* connected by zero-dimensional *joints* in $\mathbb{R}^d$. Each pair of bars connected by a joint is allowed to move continuously so that its relative motion is a d-dimensional rotation around the joint. For each joint, we define its *infinitesimal motion* as a d-dimensional velocity vector. Each bar imposes a linear constraint on the infinitesimal motions of its two end joints (for more detail, see [7]). We thus have a homogeneous system of $\{\#\text{ bars}\}$ linear equations with $d \times \{\#\text{ joints}\}$ variables. The framework is called *infinitesimally rigid* if such a system admits the D-dimensional solution space, where D is the degree of freedom of a rigid body in $\mathbb{R}^d$, i.e., $\binom{d+1}{2}$. Here the coefficient matrix of such a system is called the *rigidity matrix*, and a framework is infinitesimally rigid if the rank of its rigidity matrix is $d \times \{\#\text{ joints}\} - D$. This implies that at least $d \times \{\#\text{ joints}\} - D$ bars are necessary for the infinitesimal rigidity of a framework, which was first formulated by Maxwell [5].

Laman [4] established the necessary and sufficient combinatorial characterization for the infinitesimal rigidity of *generic* two-dimensional bar-joint frameworks. A framework is called generic if the rank of its rigidity matrix and its row-induced submatrices take the maximum values over all frameworks which are

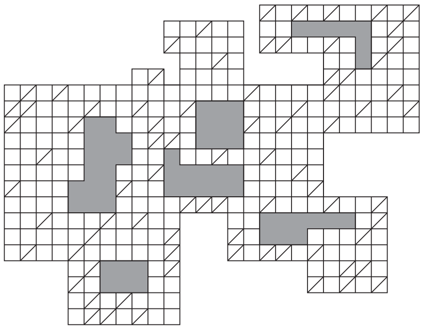


Figure 1: A square-grid framework with holes

the same topologically (also see [7]). However, combinatorial characterization for the case when $d \geq 3$ has not been found yet.

Even in two-dimension, where there are many applications in the real world (e.g., civil engineering or architectural engineering), it is worth to study non-generic frameworks. Based on this, we study *square-grid frameworks* in this paper. A square-grid framework is a connected two-dimensional bar-joint framework consisting of a union of unit grid squares, where some contain diagonal braces (see Figure 1). We assume that (i) all joints are located at the integer grid, (ii) the area of every hole is at least two, and (iii) any two of the outer face and holes of a framework do not share any joints.

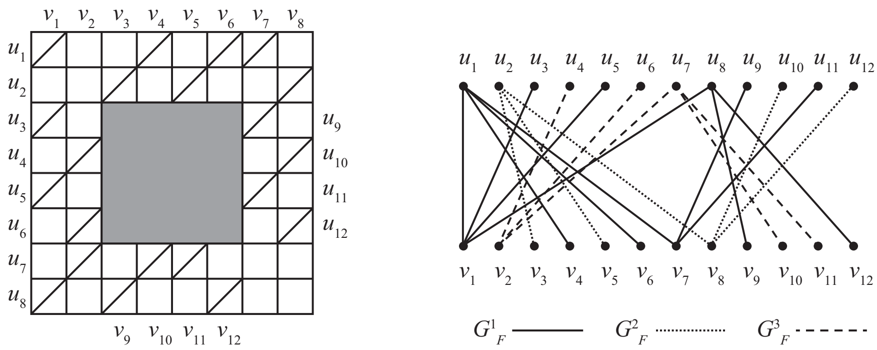
In the literature, the rigidity of square-grid frameworks was first studied by Bolker and Crapo [1]. They gave a combinatorial characterization for the infinitesimal rigidity of frameworks with no holes (which will be shown at Theorem 1). Also Gáspár, Radics and Recski [2] studied the case with holes and provided a method to determine the infinitesimal rigidity of a framework, which is faster than computing the rank of its rigidity matrix (see Theorem 2). Ito, Kobayashi, Higashikawa, Katoh, Poon and Saumell [3] have recently proposed an algorithm for the bracing problem: given a square-grid framework with holes in which there is no brace, the objective is to add the minimum number of braces which makes the framework infinitesimally rigid.

A square-grid framework is called *brace-minimally rigid* if the framework is infinitesimally rigid and removing any brace makes the framework infinitesimally flexible. Then the characterization by [1] immediately provides a necessary and sufficient condition for the brace-minimal rigidity of a framework with no holes, however the results by [2, 3] do not provide such an explicit condition for the case with holes (though the result by [2] provides a brute-force way to check the brace-minimal rigidity of a square-grid framework with holes, see just after Theorem 3). In this paper, we give the first necessary and sufficient condition for the brace-minimal rigidity of a square-grid framework with holes.

2 Preliminaries

In a square-grid framework, there are two types of bars (other than braces), that is, *horizontal-bars* and *vertical-bars* (for short, h-bars and v-bars). We define an *h-strip* (resp. a *v-strip*) as a maximal set of horizontally (resp. vertically) consecutive grid squares. Let us give all h-strips and v-strips indices, respectively. If the i-th h-strip and the j-th v-strip intersect each other at a unit grid square, we call it square (i, j). For each h-bar (resp. v-bar), we define an *infinitesimal rotation* as the difference between the vertical (resp. horizontal) components of its two end joint's infinitesimal motions. As observed in [1, 2], infinitesimal rotations of all v-bars (resp. h-bars) in an h-strip (resp. a v-strip) are the same, called an infinitesimal rotation of the h-strip (resp. v-strip). In addition, if square (i, j) is braced, infinitesimal rotations of the i-th h-strip and the j-th v-strip are the same. In the subsequent discussion, we use "rigid" and "rotation" to denote "infinitesimally rigid" and "infinitesimal rotation", respectively.

Given a square-grid framework F, let G_F denote a bipartite graph consisting of the vertex sets U_F, V_F and the edge set E_F such that $u_i \in U_F$ corresponds to the i-th h-strip in F, $v_j \in V_F$ corresponds to the



(a) F: a framework with a hole

(b) G_F: a bipartite graph consisting of connected components G_F^1, G_F^2 and G_F^3

Figure 2: Illustrations of a square-grid framework F and the corresponding bipartite graph G_F

j-th v-strip in F, and for $u_i \in U_F$ and $v_j \in V_F$ an edge $(u_i, v_j) \in E_F$ if square (i, j) is braced in F (see Figure 2). In the following, if $u \in U_F$ (resp. $v \in V_F$) corresponds to an h-strip (resp. v-strip) in F, we treat u (resp. v) as the h-strip (resp. v-strip) itself. Also, if $e \in E_F$ corresponds to a brace in F, we treat e as the brace itself. Let us observe the theorem by Bolker et al. [1].

Theorem 1 [1] *A square-grid framework F with no holes is rigid if and only if G_F is connected.*

Suppose that there are q connected components of G_F, say $G_F^1, \ldots, G_F^q$ (see Figure 2(b)). For some integer $l \in \{1, \ldots, q\}$, let U_F^l, V_F^l and E_F^l denote subsets of U_F, V_F and E_F such that $G_F^l = (U_F^l, V_F^l, E_F^l)$, respectively. We say that the i-th h-strip (resp. the j-th v-strip) belongs to G_F^l if $u_i \in U_F^l$ (resp. $v_j \in V_F^l$). If $u_i \in U_F^l$, $v_j \in V_F^l$ and square (i, j) is braced in F, we also say that the brace at square (i, j) belongs to G_F^l.

We introduce the *hole matrix* of F, which was first defined in [2]. Suppose that there are p holes in F, say *holes* $1, \ldots, p$. Let us focus on hole $k \in \{1, \ldots, p\}$. We define a *left h-strip* for hole k as an h-strip such that the rightmost v-bar in the h-strip is on the boundary of hole k. Similarly, we also define a *right h-strip*, an *upper v-strip* and a *lower v-strip* for hole k (see Figure 3). Then, as observed in [2], we can see that for any hole $k \in \{1, \ldots, p\}$

$$\sum\{\text{rotations of left h-strips}\} - \sum\{\text{rotations of right h-strips}\} = 0, \quad \text{and} \tag{1}$$

$$\sum\{\text{rotations of upper v-strips}\} - \sum\{\text{rotations of lower v-strips}\} = 0. \tag{2}$$

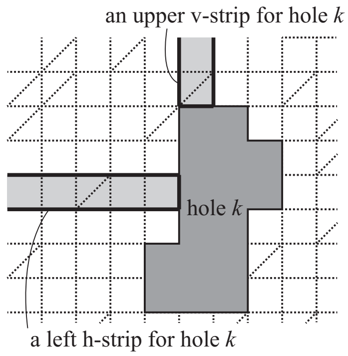


Figure 3: Illustration of a left h-strip and an upper v-strip for hole k

Note that rotations of all h-strips and v-strips belonging to a connected component are the same. Let ω_l denote a rotation of G_F^l. For integers $k \in \{1, \ldots, p\}$ and $l \in \{1, \ldots, q\}$, let α_{kl}^+ (resp. α_{kl}^-, β_{kl}^+ and β_{kl}^-) be the number of left h-strips (resp. right h-strips, upper v-strips and lower h-strips) for hole k belonging to G_F^l. Then, equations (1) and (2) can be written as

$$\sum_{l \in \{1,\ldots,q\}} (\alpha_{kl}^+ - \alpha_{kl}^-)\omega_l = 0 \quad \forall k \in \{1, \ldots, p\}, \quad \text{and} \tag{3}$$

$$\sum_{l \in \{1,\ldots,q\}} (\beta_{kl}^+ - \beta_{kl}^-)\omega_l = 0 \quad \forall k \in \{1, \ldots, p\}. \tag{4}$$

Letting $\alpha_{kl} = \alpha_{kl}^+ - \alpha_{kl}^-$ and $\beta_{kl} = \beta_{kl}^+ - \beta_{kl}^-$, we obtain a system of the above equations as

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1q} \\ \beta_{11} & \beta_{12} & \cdots & \beta_{1q} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{p1} & \beta_{p2} & \cdots & \beta_{pq} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5}$$

We call the coefficient matrix in (5) the hole matrix of F, denoted by H_F. Let us observe the theorem by Gáspár et al. [2].

Theorem 2 [2] *Given a square-grid framework F with p holes, suppose that G_F consists of q connected components. Then F is rigid if and only if the rank of H_F is $q - 1$.*

We now state the main theorem below, which will be proved in Section 3.

Theorem 3 *Given a square-grid framework F with p holes, suppose that G_F consists of q connected components. Then F is brace-minimally rigid if and only if (a) the rank of H_F is $q - 1$, (b) G_F is a spanning forest, and (c) $q = 2p + 1$.*

Note that applying Theorem 2, we can also determine the brace-minimal rigidity of F as follows: Check the rigidity of F and $F - e$ for every brace $e \in E_F$. If F is rigid but $F - e$ is not rigid for any $e \in E_F$, then F is brace-minimally rigid. However, in this way, we need to carry out rank calculations $O(|E_F|)$ times, whereas our result in Theorem 3 provides a much faster way with just one rank calculation.

3 Proof of Theorem 3

We prove the if part and the only if part of Theorem 3 in Sections 3.1 and 3.2, respectively.

3.1 Proof of the if part

Assume that conditions (a), (b) and (c) are satisfied, i.e., the rank of H_F is $2p$ and G_F is a spanning forest with $2p + 1$ connected components. We immediately see that F is rigid by Theorem 2. Consider removing a brace from F. Let F' be the resulting framework. Then $G_{F'}$ is a spanning forest with $2p + 2$ connected components. On the other hand, the rank of $H_{F'}$ is at most $2p$ since the number of rows in $H_{F'}$ is $2p$. Therefore by Theorem 2, F' is no longer rigid, which means that F is brace-minimally rigid. This completes the proof of the if part of Theorem 3.

3.2 Proof of the only if part

We show the contrapositive of the only if part: "F is not brace-minimally rigid if one of conditions (a), (b), and (c) is not satisfied."

Case 1: Condition (a) is not satisfied, i.e., the rank of H_F is not $q-1$. Note that the rank of H_F is at most $q-1$ since the sum of all q columns in H_F is zero. Thus, in this case the rank of H_F is less than $q-1$, which means that F is not rigid by Theorem 2.
Case 2: Condition (b) is not satisfied, i.e., G_F has a cycle. In this case we remove a brace corresponding to an edge on the cycle, and let F' be the resulting framework. Since $H_{F'} = H_F$, if F' is rigid, F is not brace-minimally rigid; otherwise, F is not rigid.
Case 3: Condition (c) is not satisfied, i.e., $q \neq 2p+1$. In this case we can assume that both of conditions (a) and (b) are satisfied. We have two subcases [Case 3A] $q > 2p+1$ and [Case 3B] $q < 2p+1$. First consider Case 3A. Recall that the rank of H_F is at most $2p$, which is less than $q-1$ in this subcase. This means that F is not rigid by Theorem 2. In Section 3.2.1, we consider the remaining subcase.

3.2.1 Case 3B

In this section, we consider the case that the rank of H_F is $q-1$ and G_F is a spanning forest with q connected components, where $q < 2p+1$. For this case, we show the existence of a redundant brace in F, i.e., there exists a brace such that the framework is still rigid even if we remove the brace from F.

Let R_F denote the set of rows in H_F. Suppose that $R_F = \{\boldsymbol{r}^{\mathrm{h}}_1, \boldsymbol{r}^{\mathrm{v}}_1, \boldsymbol{r}^{\mathrm{h}}_2, \ldots, \boldsymbol{r}^{\mathrm{v}}_p\}$, where

$$\boldsymbol{r}^{\mathrm{h}}_k = \begin{bmatrix} \alpha_{k1} & \alpha_{k2} & \cdots & \alpha_{kq} \end{bmatrix} \quad \forall k \in \{1, \ldots, p\}, \quad \text{and} \tag{6}$$

$$\boldsymbol{r}^{\mathrm{v}}_k = \begin{bmatrix} \beta_{k1} & \beta_{k2} & \cdots & \beta_{kq} \end{bmatrix} \quad \forall k \in \{1, \ldots, p\}. \tag{7}$$

We first determine a maximal independent set $B \subseteq R_F$ and a row $\boldsymbol{r}^* \in R_F \setminus B$ using the following procedure. Note that since $|B| = q-1 < 2p = |R_F|$, $R_F \setminus B \neq \emptyset$.

1. Choose a maximal independent set $B \subseteq R_F$ and a row $\boldsymbol{r}^* \in R_F \setminus B$ arbitrarily. Suppose $\boldsymbol{r}^* = \boldsymbol{r}^{\mathrm{h}}_{k^*}$ with an integer $k^* \in \{1, \ldots, p\}$.
2. If there exists a left h-strip for hole k^* with at least one brace, the chosen B and $\boldsymbol{r}^*$ are valid; otherwise go to 3.
3. In this case, the leftmost v-bar in every left h-strip for hole k^* is on the boundary of another hole by Lemma 5 (as shown in Figure 4). Call such a hole a *left hole* for hole k^*. In the left holes for hole k^*, if there exists a hole k' such that $\boldsymbol{r}^{\mathrm{h}}_{k'} \in R_F \setminus B$, set $k^* \leftarrow k'$ and $\boldsymbol{r}^* \leftarrow \boldsymbol{r}^{\mathrm{h}}_{k'}$, and go to 2; otherwise go to 4.
4. In this case, any left hole for hole k^*, say hole k', satisfies $\boldsymbol{r}^{\mathrm{h}}_{k'} \in B$. Since $B \setminus \{\boldsymbol{r}^{\mathrm{h}}_{k'}\} \cup \{\boldsymbol{r}^{\mathrm{h}}_{k*}\}$ is also independent by Lemma 6, set $B \leftarrow B \setminus \{\boldsymbol{r}^{\mathrm{h}}_{k'}\} \cup \{\boldsymbol{r}^{\mathrm{h}}_{k*}\}$, $k^* \leftarrow k'$ and $\boldsymbol{r}^* \leftarrow \boldsymbol{r}^{\mathrm{h}}_{k'}$, and go to 2.

Using the above procedure, we obtain $B \subseteq R_F$ and a row $\boldsymbol{r}^* = \boldsymbol{r}^{\mathrm{h}}_{k^*} \in R_F \setminus B$ such that there exists a left h-strip for hole k^* with at least one brace. If $\boldsymbol{r}^* = \boldsymbol{r}^{\mathrm{v}}_{k^*}$ with an integer $k^* \in \{1, \ldots, p\}$ at the first step, we apply the similar procedure, changing "left", "h-" and "v-" to "upper", "v-" and "h-", respectively.

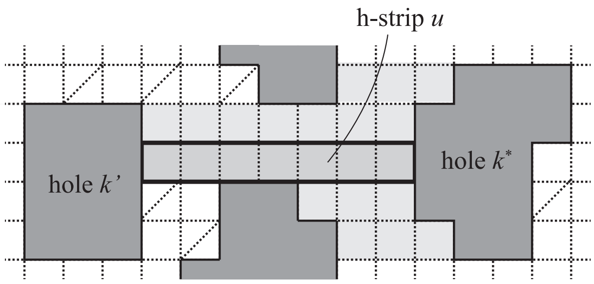


Figure 4: An h-strip u with no brace which is a left h-strip for hole k^* and a right h-strip for hole k' (the light gray part represents all the left h-strips for hole k^* where there is no brace)

Lemma 4 *For a maximal independent set $B \subseteq R_F$ and a row $\boldsymbol{r}^* = \boldsymbol{r}^{\mathrm{h}}_{k^*}$* (resp. $\boldsymbol{r}^{\mathrm{v}}_{k^*}$) $\in R_F \setminus B$ *with $k^* \in \{1,\ldots,p\}$ which are determined by the above procedure, there exists a left h-strip* (resp. *an upper v-strip*) *for hole k^* with at least one brace.*

We prove Lemmas 5 and 6 which the third and forth steps of procedure are based on, respectively.

Lemma 5 *Given a maximal independent set $B \subseteq R_F$ and a row $\boldsymbol{r}^{\mathrm{h}}_{k^*} \in R_F \setminus B$, suppose that there exists an h-strip $u \in U_F$ with no brace which is a left h-strip for hole k^* (see Figure 4). Then, the leftmost v-bar in h-strip u is on the boundary of another hole.*

Proof: Suppose that the leftmost v-bar in h-strip u is on the boundary of the outer face. In q connected components of G_F, there exists G^l_F with $l \in \{1,\ldots,q\}$ which consists of only u. Looking at the l-th column, $\alpha_{k^*l} = 1$ and all the other entries are zero. This implies that $B \cup \{\boldsymbol{r}^{\mathrm{h}}_{k^*}\}$ is independent, which contradicts the maximality of B. □

Lemma 6 *Given a maximal independent set $B \subseteq R_F$ and a row $\boldsymbol{r}^{\mathrm{h}}_{k^*} \in R_F \setminus B$, suppose that there exists an h-strip $u \in U_F$ with no brace which is a left h-strip for hole k^* and a right h-strip for hole k' (see Figure 4). Then, $B \setminus \{\boldsymbol{r}^{\mathrm{h}}_{k'}\} \cup \{\boldsymbol{r}^{\mathrm{h}}_{k*}\}$ is independent.*

Proof: In q connected components of G_F, there exists G^l_F with $l \in \{1,\ldots,q\}$ which consists of only u. Then, looking at the l-th column, $\alpha_{k^*l} = 1$, $\alpha_{k'l} = -1$ and all the other entries are zero. This implies that $B \setminus \{\boldsymbol{r}^{\mathrm{h}}_{k'}\} \cup \{\boldsymbol{r}^{\mathrm{h}}_{k*}\}$ is independent. □

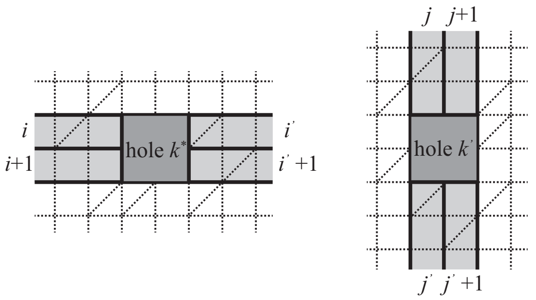
Suppose that $B \subseteq R_F$ and $\boldsymbol{r}^* \in R_F \setminus B$ have been determined by the above procedure. Also consider the unique minimal dependent set $C \subseteq B \cup \{\boldsymbol{r}^*\}$. We then observe that $\boldsymbol{r}^* \in C$ and if $\boldsymbol{r}^* = \boldsymbol{r}^{\mathrm{h}}_{k^*}$ (resp. $\boldsymbol{r}^{\mathrm{v}}_{k^*}$) with $k^* \in \{1,\ldots,p\}$, there exists a left h-strip (resp. upper v-strip) for hole k^* with at least one brace by Lemma 4. Let $G^{l^*}_F$ be the connected component of G_F which such the braced left h-strip (resp. upper v-strip) belongs to. Recall that G_F forms a spanning forest in Case 3B, thus $G^{l^*}_F$ forms a tree. Let $G^{l^*}_F(C)$ denote the minimal subtree of $G^{l^*}_F$ including all vertices in $\bigcup_{\boldsymbol{r}\in C}\{U^{l^*}_F(\boldsymbol{r}) \cup V^{l^*}_F(\boldsymbol{r})\}$, where

$$U^{l^*}_F(\boldsymbol{r}) = \begin{cases} \{u \in U^{l^*}_F \mid u \text{ is a left or right h-strip for hole } k\} & \text{if } \boldsymbol{r} = \boldsymbol{r}^{\mathrm{h}}_k \text{ with } k \in \{1,\ldots,p\} \\ \emptyset & \text{if } \boldsymbol{r} = \boldsymbol{r}^{\mathrm{v}}_k \text{ with } k \in \{1,\ldots,p\}, \end{cases} \tag{8}$$

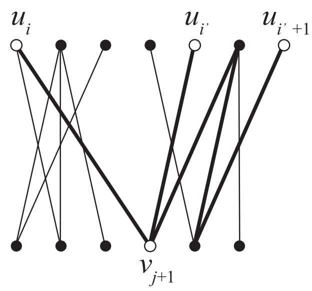
$$V^{l^*}_F(\boldsymbol{r}) = \begin{cases} \emptyset & \text{if } \boldsymbol{r} = \boldsymbol{r}^{\mathrm{h}}_k \text{ with } k \in \{1,\ldots,p\} \\ \{v \in V^{l^*}_F \mid v \text{ is an upper or lower v-strip for hole } k\} & \text{if } \boldsymbol{r} = \boldsymbol{r}^{\mathrm{v}}_k \text{ with } k \in \{1,\ldots,p\}. \end{cases} \tag{9}$$

We notice that $G^{l^*}_F$ includes at least one brace (i.e., $E^{l^*}_F \neq \emptyset$) and $G^{l^*}_F(C)$ includes at least one vertex.

Let us consider an example shown in Figure 5. Suppose $C = \{\boldsymbol{r}^{\mathrm{h}}_{k^*}(= \boldsymbol{r}^*), \boldsymbol{r}^{\mathrm{v}}_{k'}\}$. We then focus on the left and right h-strips for hole k^* and the upper and lower v-strips for hole k'. As shown in Figure 5(a),

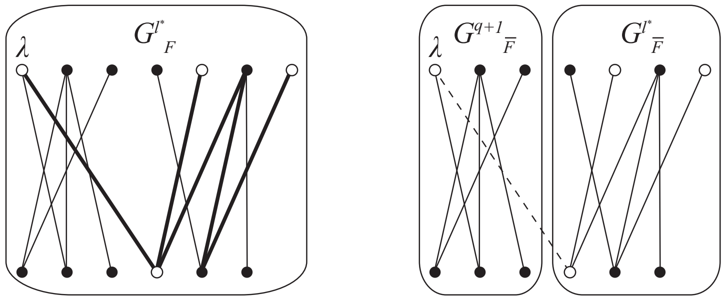


(a) the left and right h-strips for hole k^* and the upper and lower v-strips for hole k'



(b) $G^{l^*}_F$

Figure 5: An example with $C = \{\boldsymbol{r}^{\mathrm{h}}_{k^*}(= \boldsymbol{r}^*), \boldsymbol{r}^{\mathrm{v}}_{k'}\}$



(a) $\lambda(= u_i$ in Figure 5(b))

(b) $G^{l^*}_{\bar{F}}$ and $G^{q+1}_{\bar{F}}$ obtained by removing the dashed edge

Figure 6: Illustrations of $G^{l^*}_F$, $G^{l^*}_{\bar{F}}$ and $G^{q+1}_{\bar{F}}$ for the same example in Figure 5

the i-th and $(i+1)$-th (resp. i'-th and $(i'+1)$-th) h-strips lie on the left (resp. right) side of hole k^* and the j-th and $(j+1)$-th (resp. j'-th and $(j'+1)$-th) v-strips lie on the upper (resp. lower) side of hole k' in this example. Here a left h-strip for hole k^*, i.e., the i-th h-strip, is given a brace, we thus determine $G^{l^*}_F$ as a component of G_F which includes vertex u_i (see Figure 5(b)). Suppose that $G^{l^*}_F$ also includes $u_{i'}$, $u_{i'+1}$ and v_{j+1} and does not include u_{i+1}, v_j, $v_{j'}$ and $v_{j'+1}$, i.e., $\bigcup_{\boldsymbol{r}\in C}\{U^{l^*}_F(\boldsymbol{r}) \cup V^{l^*}_F(\boldsymbol{r})\} = \{u_i, u_{i'}, u_{i'+1}, v_{j+1}\}$. In Figure 5(b), white circles represent vertices in $\bigcup_{\boldsymbol{r}\in C}\{U^{l^*}_F(\boldsymbol{r}) \cup V^{l^*}_F(\boldsymbol{r})\}$ and heavy lines represent $G^{l^*}_F(C)$.

In what follows, we show the existence of a redundant brace in $G^{l^*}_F$. Note that every leaf of $G^{l^*}_F(C)$ must be a vertex in $\bigcup_{\boldsymbol{r}\in C}\{U^{l^*}_F(\boldsymbol{r}) \cup V^{l^*}_F(\boldsymbol{r})\}$ (see Figure 6(a)). Choose one of those leaves of $G^{l^*}_F(C)$, say λ. If $G^{l^*}_F(C)$ includes two or more vertices, remove a brace in $G^{l^*}_F(C)$ incident to λ (see Figure 6(b)); otherwise remove an arbitrary brace in $G^{l^*}_F$. Let $\bar{F}$ be the resulting whole framework. Then, for $l \in \{1,\ldots,q\}\setminus\{l^*\}$, the l-th connected component remains in $G_{\bar{F}}$ without any change, i.e., $G^l_{\bar{F}} = G^l_F$. Only $G^{l^*}_F$ is separated into two components $G^{l^*}_{\bar{F}}$ and $G^{q+1}_{\bar{F}}$ so that $G^{q+1}_{\bar{F}}$ includes λ and $G^{l^*}_{\bar{F}}$ does not.

Next let us see hole matrix $H_{\bar{F}}$ consisting of rows $R_{\bar{F}}$. Note that $|R_{\bar{F}}| = |R_F| = 2p$ and each row in $R_{\bar{F}}$ consists of $q+1$ entries. For integers $k \in \{1,\ldots,p\}$ and $l \in \{1,\ldots,q+1\}$, let $\bar{\alpha}^+_{kl}$ (resp. $\bar{\alpha}^-_{kl}$, $\bar{\beta}^+_{kl}$ and $\bar{\beta}^-_{kl}$) be the number of left h-strips (resp. right h-strips, upper v-strips and lower h-strips) for hole k belonging to $G^l_{\bar{F}}$. Let $\bar{\alpha}_{kl} = \bar{\alpha}^+_{kl} - \bar{\alpha}^-_{kl}$ and $\bar{\beta}_{kl} = \bar{\beta}^+_{kl} - \bar{\beta}^-_{kl}$, respectively. For the h-th row $\boldsymbol{r} \in R_F$ with $h \in \{1,\ldots,2p\}$, let $\rho(\boldsymbol{r})$ denote the h-th row in $R_{\bar{F}}$. Then $R_{\bar{F}} = \{\rho(\boldsymbol{r}^{\mathrm{h}}_1), \rho(\boldsymbol{r}^{\mathrm{v}}_1), \rho(\boldsymbol{r}^{\mathrm{h}}_2), \ldots, \rho(\boldsymbol{r}^{\mathrm{v}}_p)\}$, where

$$\rho(\boldsymbol{r}^{\mathrm{h}}_k) = \begin{bmatrix} \bar{\alpha}_{k1} & \bar{\alpha}_{k2} & \cdots & \bar{\alpha}_{kq} & \bar{\alpha}_{k,q+1} \end{bmatrix} \quad \forall k \in \{1,\ldots,p\}, \quad \text{and} \tag{10}$$

$$\rho(\boldsymbol{r}^{\mathrm{v}}_k) = \begin{bmatrix} \bar{\beta}_{k1} & \bar{\beta}_{k2} & \cdots & \bar{\beta}_{kq} & \bar{\beta}_{k,q+1} \end{bmatrix} \quad \forall k \in \{1,\ldots,p\}. \tag{11}$$

In the following, for $R' \subseteq R_F$, we abuse the notation $\rho(R')$ to denote the subset of $R_{\bar{F}}$, $\{\rho(\boldsymbol{r}) \mid \boldsymbol{r} \in R'\}$. In addition, we use the notation $\varepsilon_l(\boldsymbol{r})$ to denote the l-th entry of row $\boldsymbol{r}$, e.g., $\varepsilon_l(\boldsymbol{r}^{\mathrm{h}}_k) = \alpha_{kl}$ and $\varepsilon_l(\rho(\boldsymbol{r}^{\mathrm{v}}_k)) = \bar{\beta}_{kl}$. Let us see the following remark.

Remark 7 *For any $\boldsymbol{r} \in R_F$ and $l \in \{1,\ldots,q\}\setminus\{l^*\}$, $\varepsilon_l(\rho(\boldsymbol{r})) = \varepsilon_l(\boldsymbol{r})$.*

Recall that λ is the leaf of $G^{l^*}_F(C)$ which is included in $G^{q+1}_{\bar{F}}$. Let $\tilde{\boldsymbol{r}}$ be a row in C such that $\lambda \in U^{l^*}_F(\tilde{\boldsymbol{r}}) \cup V^{l^*}_F(\tilde{\boldsymbol{r}})$. We then show the following lemma.

Lemma 8 *(i) For $\boldsymbol{r} \in R_F$, $\varepsilon_{l^*}(\rho(\boldsymbol{r})) + \varepsilon_{q+1}(\rho(\boldsymbol{r})) = \varepsilon_{l^*}(\boldsymbol{r})$. (ii) $\varepsilon_{l^*}(\rho(\tilde{\boldsymbol{r}})) = \varepsilon_{l^*}(\tilde{\boldsymbol{r}}) - \delta$ and $\varepsilon_{q+1}(\rho(\tilde{\boldsymbol{r}})) = \delta$, where $\delta = -1$ or 1. (iii) For $\boldsymbol{r} \in C\setminus\{\tilde{\boldsymbol{r}}\}$, $\varepsilon_{l^*}(\rho(\boldsymbol{r})) = \varepsilon_{l^*}(\boldsymbol{r})$ and $\varepsilon_{q+1}(\rho(\boldsymbol{r})) = 0$.*

Proof: (i) immediately follows the fact that $G^{l^*}_F$ is separated into $G^{l^*}_{\bar{F}}$ and $G^{q+1}_{\bar{F}}$. Suppose that $\tilde{\boldsymbol{r}} = \boldsymbol{r}^{\mathrm{h}}_{\tilde{k}}$ with $\tilde{k} \in \{1,\ldots,p\}$. Then λ is a left or right h-strip for hole $\tilde{k}$. If λ is a left h-strip for hole $\tilde{k}$, we have $\bar{\alpha}^+_{\tilde{k}l^*} = \alpha^+_{\tilde{k}l^*} - 1$, $\bar{\alpha}^-_{\tilde{k}l^*} = \alpha^-_{\tilde{k}l^*}$, $\bar{\alpha}^+_{\tilde{k},q+1} = 1$, and $\bar{\alpha}^-_{\tilde{k},q+1} = 0$, i.e., $\bar{\alpha}_{\tilde{k}l^*} = \alpha_{\tilde{k}l^*} - 1$ and $\bar{\alpha}_{\tilde{k},q+1} = 1$; otherwise $\bar{\alpha}_{\tilde{k}l^*} = \alpha_{\tilde{k}l^*} + 1$ and $\bar{\alpha}_{\tilde{k},q+1} = -1$. Similarly, for the case that $\tilde{\boldsymbol{r}} = \boldsymbol{r}^{\mathrm{v}}_{\tilde{k}}$ with $\tilde{k} \in \{1,\ldots,p\}$, we can prove

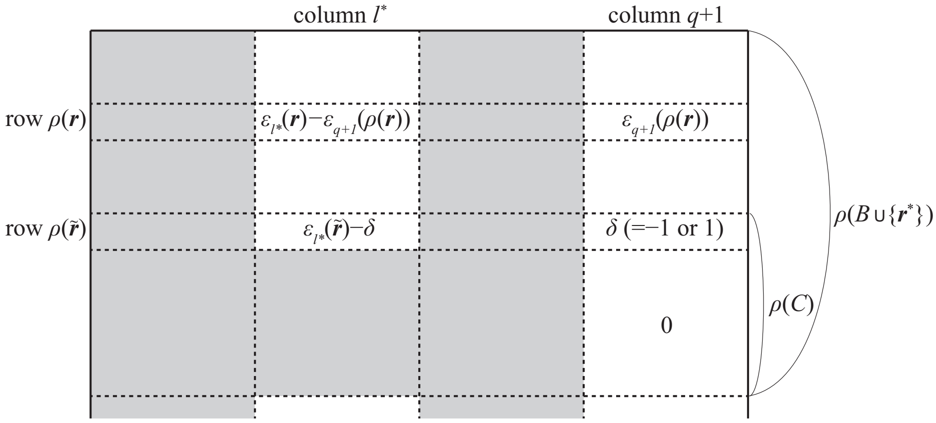


Figure 7: $H_{\bar{F}}$ (each entry in the dark part is the same as in H_F)

$\bar{\beta}_{\tilde{k}l^*} = \beta_{\tilde{k}l^*} \mp 1$ and $\bar{\beta}_{\tilde{k},q+1} = \pm 1$, thus (ii) is proved. For $\boldsymbol{r}_k^{\mathrm{h}}$ (resp. $\boldsymbol{r}_k^{\mathrm{v}}$) $\in C \setminus \{\tilde{\boldsymbol{r}}\}$, any left or right h-strip (resp. upper or lower v-strip) for hole k belonging to $G_F^{l^*}$ remains in $G_{\bar{F}}^{l^*}$, thus (iii) holds. $\square$

We represent the statements of Remark 7 and Lemma 8 as an illustration in Figure 7.

In the rest of this section, we show that $\bar{F}$ is rigid, i.e., F is not brace-minimally rigid. Let us first confirm the following remark.

Remark 9 *$B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}$ is independent.*

We then show the following lemma.

Lemma 10 *$\rho(B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\})$ is independent.*

Proof: We prove by contradiction. Suppose that $\rho(B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\})$ is dependent:

$$\sum_{\boldsymbol{r}\in B\cup\{\boldsymbol{r}^*\}\setminus\{\tilde{\boldsymbol{r}}\}} \mu_{\boldsymbol{r}} \cdot \rho(\boldsymbol{r}) = \boldsymbol{0}, \tag{12}$$

where $\mu_{\boldsymbol{r}}$ is a real number such that $\mu_{\boldsymbol{r}} \neq 0$ $\exists \boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}$. The left-hand side of (12) can be represented as

$$\sum_{\boldsymbol{r}\in B\cup\{\boldsymbol{r}^*\}\setminus\{\tilde{\boldsymbol{r}}\}} \mu_{\boldsymbol{r}} \cdot \left[\ \varepsilon_1(\rho(\boldsymbol{r})) \quad \cdots \quad \varepsilon_{l^*}(\rho(\boldsymbol{r})) \quad \cdots \quad \varepsilon_q(\rho(\boldsymbol{r})) \quad \varepsilon_{q+1}(\rho(\boldsymbol{r}))\ \right]. \tag{13}$$

By Remark 7 and Lemma 8(ii), (13) is equal to

$$\sum_{\boldsymbol{r}\in B\cup\{\boldsymbol{r}^*\}\setminus\{\tilde{\boldsymbol{r}}\}} \mu_{\boldsymbol{r}} \cdot \left[\ \varepsilon_1(\boldsymbol{r}) \quad \cdots \quad \varepsilon_{l^*}(\boldsymbol{r}) \quad \cdots \quad \varepsilon_q(\boldsymbol{r}) \quad 0\ \right]$$
$$+ \sum_{\boldsymbol{r}\in B\cup\{\boldsymbol{r}^*\}\setminus\{\tilde{\boldsymbol{r}}\}} \mu_{\boldsymbol{r}} \cdot \left[\ 0 \quad \cdots \quad -\varepsilon_{q+1}(\rho(\boldsymbol{r})) \quad \cdots \quad 0 \quad \varepsilon_{q+1}(\rho(\boldsymbol{r}))\ \right]. \tag{14}$$

Looking at the $(q+1)$-th entries in (14), the sum of those entries is zero by (12), i.e.,

$$\sum_{\boldsymbol{r}\in B\cup\{\boldsymbol{r}^*\}\setminus\{\tilde{\boldsymbol{r}}\}} \mu_{\boldsymbol{r}} \cdot \varepsilon_{q+1}(\rho(\boldsymbol{r})) = 0. \tag{15}$$

This means that the second summation term in (14) is a zero vector, and therefore the first term is also a zero vector. We thus have

$$\sum_{\boldsymbol{r}\in B\cup\{\boldsymbol{r}^*\}\setminus\{\tilde{\boldsymbol{r}}\}} \mu_{\boldsymbol{r}} \cdot \boldsymbol{r} = \boldsymbol{0}, \tag{16}$$

which contradicts the independency of $B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}$ shown in Remark 9. $\square$

The following lemma implies the rigidity of $\bar{F}$.

Lemma 11 *$\rho(B \cup \{\boldsymbol{r}^*\})$ is independent.*

PROOF: We prove by contradiction. Suppose that $\rho(B \cup \{\boldsymbol{r}^*\})$ is dependent. Since $\rho(B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\})$ is independent by Lemma 10, any dependent subset of $\rho(B \cup \{\boldsymbol{r}^*\})$ includes $\rho(\tilde{\boldsymbol{r}})$. Thus, $\rho(\tilde{\boldsymbol{r}})$ is represented as a linear combination of rows in $\rho(B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\})$:

$$\rho(\tilde{\boldsymbol{r}}) = \sum_{\boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}} \phi_{\boldsymbol{r}} \cdot \rho(\boldsymbol{r}), \tag{17}$$

where $\phi_{\boldsymbol{r}}$ is a real number such that $\phi_{\boldsymbol{r}} \neq 0 \; \exists \boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}$. Equation (17) can be deformed as

$$\begin{aligned} &\left[\; \varepsilon_1(\rho(\tilde{\boldsymbol{r}})) \quad \cdots \quad \varepsilon_{l^*}(\rho(\tilde{\boldsymbol{r}})) \quad \cdots \quad \varepsilon_q(\rho(\tilde{\boldsymbol{r}})) \quad \varepsilon_{q+1}(\rho(\tilde{\boldsymbol{r}})) \;\right] \\ &\qquad = \sum_{\boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}} \phi_{\boldsymbol{r}} \cdot \left[\; \varepsilon_1(\rho(\boldsymbol{r})) \quad \cdots \quad \varepsilon_{l^*}(\rho(\boldsymbol{r})) \quad \cdots \quad \varepsilon_q(\rho(\boldsymbol{r})) \quad \varepsilon_{q+1}(\rho(\boldsymbol{r})) \;\right]. \end{aligned} \tag{18}$$

By Remark 7 and Lemma 8(ii), the left-hand side of (18) can be represented as

$$\left[\; \varepsilon_1(\tilde{\boldsymbol{r}}) \quad \cdots \quad \varepsilon_{l^*}(\tilde{\boldsymbol{r}}) \quad \cdots \quad \varepsilon_q(\tilde{\boldsymbol{r}}) \quad 0 \;\right] + \left[\; 0 \quad \cdots \quad -\delta \quad \cdots \quad 0 \quad \delta \;\right], \tag{19}$$

where $\delta = -1$ or 1. Similarly, by Remark 7 and Lemma 8(i), the right-hand side of (18) can be represented as

$$\begin{aligned} &\sum_{\boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}} \phi_{\boldsymbol{r}} \cdot \left[\; \varepsilon_1(\boldsymbol{r}) \quad \cdots \quad \varepsilon_{l^*}(\boldsymbol{r}) \quad \cdots \quad \varepsilon_q(\boldsymbol{r}) \quad 0 \;\right] \\ &\qquad + \sum_{\boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}} \phi_{\boldsymbol{r}} \cdot \left[\; 0 \quad \cdots \quad -\varepsilon_{q+1}(\rho(\boldsymbol{r})) \quad \cdots \quad 0 \quad \varepsilon_{q+1}(\rho(\boldsymbol{r})) \;\right]. \end{aligned} \tag{20}$$

Looking at the $(q+1)$-th entries in (18), (19) and (20), we obtain

$$\delta = \sum_{\boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}} \phi_{\boldsymbol{r}} \cdot \varepsilon_{q+1}(\rho(\boldsymbol{r})), \tag{21}$$

which means that the second terms in (19) and (20) are equivalent, and therefore the first terms are also equivalent. We thus have

$$\tilde{\boldsymbol{r}} = \sum_{\boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}} \phi_{\boldsymbol{r}} \cdot \boldsymbol{r}. \tag{22}$$

Recall that $B \cup \{\boldsymbol{r}^*\}$ includes the unique minimal dependent set of rows, which is C, and $\tilde{\boldsymbol{r}} \in C$. Because of the uniqueness and the minimality of C, $\phi_{\boldsymbol{r}}$ is uniquely determined for $\boldsymbol{r} \in B \cup \{\boldsymbol{r}^*\} \setminus \{\tilde{\boldsymbol{r}}\}$ such that

$$\phi_{\boldsymbol{r}} \neq 0 \qquad \forall \boldsymbol{r} \in C \setminus \{\tilde{\boldsymbol{r}}\}, \quad \text{and} \tag{23}$$

$$\phi_{\boldsymbol{r}} = 0 \qquad \forall \boldsymbol{r} \in B \setminus C. \tag{24}$$

By (23) and (24), equation (21) can be deformed as

$$\delta = \sum_{\boldsymbol{r} \in C \setminus \{\tilde{\boldsymbol{r}}\}} \phi_{\boldsymbol{r}} \cdot \varepsilon_{q+1}(\rho(\boldsymbol{r})). \tag{25}$$

Here $\varepsilon_{q+1}(\rho(\boldsymbol{r})) = 0$ for any $\boldsymbol{r} \in C \setminus \{\tilde{\boldsymbol{r}}\}$ by Lemma 8(iii), thus the right-hand side of (25) is equal to zero, whereas $\delta = -1$ or 1, contradiction. This concludes the proof. $\square$

We now observe that the rank of $H_{\bar{F}}$ is at least q since $\rho(B \cup \{\boldsymbol{r}^*\})$ consists of q independent rows by Lemma 11. Note that the rank of $H_{\bar{F}}$ is at most q since the sum of all $q+1$ columns in $H_{\bar{F}}$ is zero, so the rank of $H_{\bar{F}}$ is q. This means that $\bar{F}$ is rigid by Theorem 2, i.e., F is not brace-minimally rigid. We thus complete the proof of Theorem 3.

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