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Characterizing redundant rigidity and redundant global rigidity of body-hinge graphs

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ABSTRACT

In this paper, we characterize the redundant rigidity and the redundant global rigidity of body-hinge graphs in $\mathbb{R}^d$ in terms of graph connectivity.
Although an efficient algorithm which determines mixed-connectivity is still not known, our result implies that both edge-redundancy for rigidity and edge-redundancy for global rigidity can be checked via efficient graph-connectivity algorithms.

1. Introduction

The aim of this paper is to characterize the redundant rigidity and the redundant global rigidity of body-hinge graphs in $\mathbb{R}^d$ in terms of graph connectivity. Graph connectivity has been extensively studied [1,13] and several previous studies had investigated the connection between rigidity and graph connectivity in the context of 2-dimensional bar and joint frameworks [3,8,15]. The motivation to study body-hinge frameworks is due to their extensive use in real-world applications such as robotics, engineering, material science and computational biology [6,22]. We now define the notion of mixed-connectivity.

Definition 1 *(Mixed-connectivity).* Let k and h be integers such that $k \geq 1$ and $h \geq 1$, respectively. A graph G is (k,h)-connected if removing any $(k-1)$ vertices from G results in a graph which is h-edge-connected.

A d-dimensional body-hinge framework is a collection of d-dimensional rigid bodies connected by revolute *hinges* (see Fig. 1 and [17,20] for further details). We say a d-dimensional body-hinge framework is *rigid* if every motion results in a framework isometric to the original one (i.e. the motion corresponds to an isometry of $\mathbb{R}^d$); such motions are called *trivial* or *rigid-body motions*. Otherwise a framework is called *flexible* [7,21]. The underlying combinatorial structure of a body-hinge framework is a multigraph $G = (V, E)$, where V and E represent a set of bodies and a set of hinges, respectively. Namely $uv \in E$ corresponds to a hinge $\mathbf{p}(uv)$ (i.e. a $(d-2)$-dimensional affine subspace) which joins the two bodies u and v. G is said to

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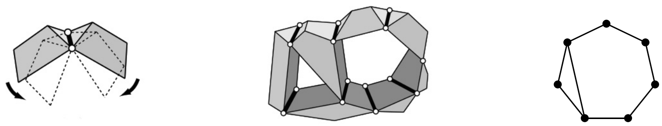

(a) (b) (c)

Fig. 1. (a) Two bodies rotating about a connecting hinge (line) in 3-space. (b) A body-hinge framework and (c) its underlying graph G.

be *realized* as a body-hinge framework $(G, \mathbf{p})$ in $\mathbb{R}^d$, and is called a *body-hinge graph*. When a body-hinge graph G can be realized as an infinitesimally rigid body-hinge framework in $\mathbb{R}^d$, G is called *rigid* [17,20]. We call a *body-hinge graph* simply a *graph*.

Proposition 1. *(See [17,20].) A graph G can be realized as a rigid body-hinge framework in $\mathbb{R}^d$ with $d \geq 2$ if and only if $(D-1)G$ contains D edge-disjoint spanning trees, where $D = \binom{d+1}{2}$ and $(D-1)G$ denotes the graph obtained from G by replacing each edge by $(D-1)$ parallel edges.*

In the following, a graph G is called *h-edge-rigid* in $\mathbb{R}^d$ if removing any $(h-1)$ edges from G results in a graph which is rigid in $\mathbb{R}^d$. The reader should keep in mind that rigidity (also h-edge rigidity and (k, h)-rigidity; see below) of a graph is ambiguous unless the underlying dimension is specified. Our definitions and results apply to any dimension d ($d \geq 2$); the dimension will be specified in the provided examples.

We now define the notion of redundant rigidity for graphs.

Definition 2 *(Redundant rigidity).* Let k and h be integers such that $k \geq 1$ and $h \geq 1$, respectively. A graph G is called (k, h)-rigid in $\mathbb{R}^d$ with $d \geq 2$ if removing any $(k-1)$ vertices from G results in a graph which is h-edge-rigid in $\mathbb{R}^d$.

Furthermore, our work has applications to global rigidity. We say that $(G, \mathbf{p})$ is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional framework which is *equivalent* to $(G, \mathbf{p})$ is *congruent* to $(G, \mathbf{p})$ (see [4] for details). A graph G is globally rigid in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. A graph G is called *h-edge-globally rigid* in $\mathbb{R}^d$ if removing any $(h-1)$ edges from G results in a graph which is globally rigid in $\mathbb{R}^d$.

We now define the notion of redundant global rigidity for graphs.

Definition 3 *(Redundant global rigidity).* Let k and h be integers such that $k \geq 1$ and $h \geq 1$, respectively. A graph G is called (k, h)-globally rigid in $\mathbb{R}^d$ with $d \geq 2$ if removing any $(k-1)$ vertices from G results in a graph which is h-edge-globally rigid in $\mathbb{R}^d$.

The main result of this paper is stated in the following theorem.

Theorem 1. *Let k and h be integers such that $k \geq 1$ and $h \geq 2$, respectively.*

(1) A graph G is (k, h)-rigid in $\mathbb{R}^2$ if and only if G is $(k, h+1)$-connected and G is (k, h)-globally rigid in $\mathbb{R}^2$ if and only if G is $(k, h+2)$-connected.

(2) For any $d \geq 3$, the following three statements are equivalent for any graph G: (i) G is (k, h)-rigid in $\mathbb{R}^d$, (ii) G is (k, h)-globally rigid in $\mathbb{R}^d$, (iii) G is $(k, h+1)$-connected.

2. Preliminaries

White and Whiteley [19] defined the infinitesimal motions of a body-hinge framework by using real vectors of length $\binom{d+1}{2}$, called *screw centers*. $(G, \mathbf{p})$ is said to be *infinitesimally rigid* if all infinitesimal motions of $(G, \mathbf{p})$ are trivial (see [5] for details). Tay [17] and Whiteley [20] independently proved that the infinitesimal rigidity of a *generic* body-hinge framework $(G, \mathbf{p})$ is determined only by its underlying graph G. A body-hinge framework is *generic* if its rigidity matrix has a maximum rank on all subgraphs [5]. 'Almost all' body-hinge realizations of G are generic in $\mathbb{R}^d$. Note that for generic frameworks, infinitesimal rigidity is equivalent to rigidity (see [20–22] for details).

Let $G = (V, E)$ be a multigraph which may contain parallel edges but no self-loops. For $X \subseteq V$, let $G[X]$ be the graph induced by X. For $X \subseteq V$, let $\delta_G(X) = \{uv \in E \mid u \in X, \ v \notin X\}$. For $X = \{v\}$, we shall omit the set brackets when describing singleton sets, e.g., $\delta_G(\{v\})$ is simply denoted by $\delta_G(v)$. A *partition* $\mathcal{P}$ of V is a collection $\{V_1, V_2, \ldots, V_m\}$ of vertex subsets for some positive integer m such that $V_i \neq \emptyset$ for $1 \leq i \leq m$, $V_i \cap V_j = \emptyset$ for any $1 \leq i, \ j \leq m, \ i \neq j$, and $\cup_{i=1}^{m} V_i = V$. Note that $\{V\}$ is a partition of V for $m = 1$. Let $\delta_G(\mathcal{P})$ denote the set of edges of G connecting distinct subsets of $\mathcal{P}$. Let $G + e = (V, E \cup \{e\})$, for any edge $e = uv$ such that $u, v \in V$, and let $G - e = (V, E \backslash \{e\})$.

In the proof in the next section we will make use of the following well-known Tutte–Nash-Williams disjoint tree proposition [14,18].

Proposition 2. *(See Tutte, Nash-Williams.) A multigraph $G = (V, E)$ contains k edge-disjoint spanning trees if and only if $|\delta_G(\mathcal{P})| \geq k(|\mathcal{P}| - 1)$ holds for every partition $\mathcal{P}$ of V.*

Katoh et al. [5] showed the following proposition.

Proposition 3. *(See Katoh et al.) Let G be a graph. Then, G is 2-edge-connected if G is rigid in $\mathbb{R}^d$ with $d \geq 2$.*

Jordán et al. [4] characterized the global rigidity of body-hinge frameworks.

Proposition 4. *(See Jordán et al.) Let G be a graph. Then G is globally rigid in $\mathbb{R}^2$ if and only if G is 3-edge-connected.*

Proposition 5. *(See Jordán et al.) Let G be a graph and $d \geq 3$. Then G is globally rigid in $\mathbb{R}^d$ if and only if $(D-1)G - f$ contains D-edge-disjoint spanning trees for any edge f of $(D-1)G$ where $D = \binom{d+1}{2}$.*

3. Proof of Theorem 1

In this section, we prove Theorem 1. For this purpose, we prove the following three statements. (a) For $d \geq 2$, G is (k,h)-rigid in $\mathbb{R}^d$ if and only if G is $(k,h+1)$-connected. (b) G is (k,h)-globally rigid in $\mathbb{R}^2$ if and only if G is $(k,h+2)$-connected. (c) For any $d \geq 3$, G is (k,h)-rigid in $\mathbb{R}^d$ if and only if G is (k,h)-globally rigid. When we summarize the above three, we will complete the proof of Theorem 1.

Proof of the only if part in (a). We show the contraposition of the only if part: "G is not (k,h)-rigid in $\mathbb{R}^d$ if G is not $(k,h+1)$-connected". There exists a vertex set $V' \subset V$ with $|V'| \leq k-1$ such that the graph G' obtained by removing $|V'|$ from G is not $(1,h+1)$-connected. Then, by Proposition 2, there exists a vertex partition $\mathcal{P}$ such that $|\mathcal{P}| = 2$ and $|\delta_{G'}(\mathcal{P})| \leq h$. Then, removing $(h-1)$ edges in $\delta_{G'}(\mathcal{P})$ from G' results in a graph G'' such that $|\delta_{G''}(\mathcal{P})| \leq 1$, which implies G'' is not rigid by Propositions 1 and 2. Thus, the original graph G is not (k,h)-rigid. □

Proof of the if part in (a). We prove by contradiction: Suppose that G is $(k,h+1)$-connected and not (k,h)-rigid in $\mathbb{R}^d$. There exists a set of $(k-1)$ vertices in G such that the graph G' which is obtained from G by removing them is not h-edge-rigid. There exists a set F of $(h-1)$ edges such that the graph G'' obtained by removing F from G' is not rigid. By Proposition 1, the graph which is obtained from G'' by replacing each edge by $(D-1)$ parallel edges does not contain D-edge-disjoint spanning trees. Then, by Proposition 2, there exists a vertex partition $\mathcal{P}$ of G' such that $|\mathcal{P}| \geq 2$ and $(D-1)|\delta_{G''}(\mathcal{P})| < D(|\mathcal{P}|-1)$. Also, $|\delta_{G'}(\mathcal{P})| \leq |\delta_{G''}(\mathcal{P})| + h - 1$ holds, thus, we have

$$|\delta_{G'}(\mathcal{P})| < \frac{D}{D-1}(|\mathcal{P}|-1) + h - 1. \tag{1}$$

On the other hand, since G' is $(1,h+1)$-connected, we have

$$|\delta_{G'}(\mathcal{P})| \geq \frac{h+1}{2}|\mathcal{P}|. \tag{2}$$

Let us evaluate the value defined as

$$value(D,h,|\mathcal{P}|) = \frac{D}{D-1}(|\mathcal{P}|-1) + h - 1 - \frac{h+1}{2}|\mathcal{P}|.$$

Since $D \geq 3$ by $d \geq 2$, $D/(D-1) \leq 3/2$ holds, thus, we have

$$\begin{aligned} value(D,h,|\mathcal{P}|) &\leq \frac{3}{2}(|\mathcal{P}|-1) + h - 1 - \frac{h+1}{2}|\mathcal{P}| \\ &= -\frac{1}{2}(h-2)(|\mathcal{P}|-2) - \frac{1}{2}. \end{aligned}$$

Then, by the condition that $h \geq 2$ and $|\mathcal{P}| \geq 2$, $value(D,h,|\mathcal{P}|) < 0$ always holds, that is,

$$\frac{D}{D-1}(|\mathcal{P}|-1) + h - 1 < \frac{h+1}{2}|\mathcal{P}|, \tag{3}$$

which contradicts (1) and (2). □

Proof of (b). By Definitions 1, 3 and Proposition 4, the proof immediately follows. □

Proof of the if part in (c). By the definition of global rigidity, this part is straightforward. □

Proof of the only if part in (c). First, we shall show the following lemma.

Lemma 1. *Let $G = (V,E)$ be a graph such that $|V| \geq 6$ and $(D-1)|E| = D(|V|-1)$ where $D = \binom{d+1}{2}$ with $d \geq 3$. Then there exist at least six vertices of degree two in G.*

Proof of Lemma 1. Let s be the number of vertices of degree two in G. Then we have

$$2|E| \geq 2s + 3(|V| - s). \tag{4}$$

By (4) and $(D-1)|E| = D(|V|-1)$, we have

$$2D(|V|-1) \geq 2(D-1)s + 3(D-1)(|V|-s) \tag{5}$$

$$s \geq \frac{D-3}{D-1}|V| + \frac{2D}{D-1}. \tag{6}$$

By $|V| \geq 6$ and $D \geq 6$, we have $(D-3)|V|/(D-1) = (1 - 2/(D-1))|V| \geq 18/5$ and $2D/(D-1) = 2 + 2/(D-1) > 2$. Then, we have $s \geq 6$. □

Then we prove by contradiction: Suppose that G is (k,h)-rigid in $\mathbb{R}^d$ and not (k,h)-globally rigid in $\mathbb{R}^d$. Let G' be a graph which is obtained from G by removing $(k-1)$ vertices and $(h-1)$ edges from G so that G' is not globally rigid. By Proposition 5, there exists a parallel edge f of $(D-1)G'$ such that $(D-1)G' - f$ does not contain D-edge-disjoint spanning trees. Then by Proposition 2, there exists a partition $\mathcal{P}$ of vertices of G' such that $|\delta_{(D-1)G'-f}(\mathcal{P})| < D(|\mathcal{P}|-1)$. Since G' is rigid, we have $(D-1)|\delta_{G'}(\mathcal{P})| \geq D(|\mathcal{P}|-1)$. By $|\delta_{(D-1)G'-f}(\mathcal{P})| = (D-1)|\delta_{G'}(\mathcal{P})| - 1$, we have $(D-1)|\delta_{G'}(\mathcal{P})| = D(|\mathcal{P}|-1)$.

By Lemma 1, there exist at least six vertex sets of $\mathcal{P}$, say $V_1, \ldots, V_p$ with $p \geq 6$, such that $|\delta_{G'}(V_i)| = 2$ for $i = 1, \ldots, p$. For any edge e of $G \backslash G'$, let us consider $G' + e$. For the same partition $\mathcal{P}$, even if e connects V_k with V_l, there remain at least four vertex sets V_i such that $|\delta_{G'+e}(V_i)| = 2$. From $G' + e$, if we remove an edge e', which is incident to such V_i, $G' + e - e'$ becomes flexible by Proposition 3. On the other hand, by (k,h)-rigidity of G, $G' + e$ should be 2-edge-rigid, contradiction. □

4. Conclusion

We characterized the redundant rigidity and the redundant global rigidity of graphs in $\mathbb{R}^d$ in terms of graph connectivity.

Our result is contrasted with the fact that the problem of augmenting a Laman graph (i.e., the graph corresponding to a minimally rigid generic bar-joint framework in 2-dimension – see [7]) to a 2-edge-rigid bar-joint graph with a minimum number of added edges is NP-hard [2]. By Theorem 1, in order to make any graph G h-edge-rigid in any dimension by adding a minimum number of edges, we can apply a polynomial time algorithm to make G $(h+1)$-edge-connected [12].

Furthermore, by Theorem 1, we can test whether a given graph is h-edge-rigid by testing whether the graph is $(h+1)$-edge-connected in $O(|E| + \min\{h|V|^2, |E||V| + |V|^2 \log|V|\})$ [10]. In particular for $h = 2$, it can be done in linear time [9,11,16]. Similar remark also holds for testing h-edge-global rigidity.

Note, this result does not imply polynomial time algorithms for testing (k,h)-rigidity and (k,h)-global rigidity which are part of future research.

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References

[1] A. Frank, Connections in Combinatorial Optimization, Oxford Lecture Series in Mathematics and Its Applications, vol. 38, Oxford University Press, 2011.
[2] A. Garcia, J. Tejel, Augmenting the rigidity of a graph in R^2, Algorithmica 59 (2011) 145–168.
[3] B. Jackson, T. Jordán, A sufficient connectivity condition for generic rigidity in the plane, Discrete Appl. Math. 157 (8) (2009) 1965–1968.
[4] T. Jordán, C. Király, S. Tanigawa, Generic global rigidity of body-hinge frameworks, EGRES Technical Reports, TR-2014-06, 2014.
[5] N. Katoh, S. Tanigawa, A proof of the molecular conjecture, Discrete Comput. Geom. 45 (2011) 647–700.
[6] L.A. Kuhn, D.J. Rader, M.F. Thorpe, Protein flexibility predictions using graph theory, Proteins 44 (2001) 150–165.
[7] G. Laman, On graphs and rigidity of plane skeletal structures, J. Eng. Math. 4 (1970) 331–340.
[8] L. Lovász, Y. Yemini, On generic rigidity in the plane, SIAM J. Algebr. Discrete Methods 3 (1) (1982) 91–98.
[9] K. Mehlhorn, A. Neumann, J. Schmidt, Certifying 3-edge-connectivity, arXiv:1211.6553v1, 2013.
[10] H. Nagamochi, T. Ibaraki, Computing edge connectivity in multigraphs and capacitated graphs, SIAM J. Discrete Math. 5 (1992) 54–66.
[11] H. Nagamochi, T. Ibaraki, A linear time algorithm for computing 3-edge-connected components in a multigraph, Jpn. J. Ind. Appl. Math. 9 (1992) 163–180.
[12] H. Nagamochi, T. Ibaraki, Augmenting edge-connectivity over the entire range in $\tilde{O}(nm)$ time, Jpn. J. Ind. Appl. Math. 30 (1999) 253–301.
[13] H. Nagamochi, T. Ibaraki, Algorithmic Aspects of Graph Connectivity, Cambridge University Press, New York, 2008, Lemma 4.8.
[14] C. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. s1 (1) (1961) 445–450.
[15] B. Servatius, H. Servatius, Rigidity, global rigidity, and graph decomposition, Eur. J. Comb. 31 (2009) 1121–1135.
[16] S. Taoka, T. Watanabe, K. Onaga, A linear-time algorithm for computing all 3-edge-connected components of a multigraph, IEICE Trans. Fundam. Electron. Commun. Comput. Sci. 75 (3) (1992) 410–424.
[17] T. Tay, Linking $(n-2)$-dimensional panels in n-space ii: $(n-2,2)$-frameworks and body and hinge structures, Graphs Comb. 5 (1) (1989) 245–273.
[18] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. Lond. Math. Soc. s36 (1961) 221.
[19] N. White, W. Whiteley, The algebraic geometry of motions of bar-and-body frameworks, SIAM J. Algebr. Discrete Methods 8 (1) (1987) 1–32.
[20] W. Whiteley, The union of matroids and the rigidity of frameworks, SIAM J. Discrete Math. 1 (2) (1988) 237–255.
[21] W. Whiteley, Some matroids from discrete applied geometry, Contemp. Math. 197 (1996) 171–311.
[22] W. Whiteley, Counting out to the flexibility of molecules, Phys. Biol. 2 (2005) 1–11.